# A note on sedimentation and consolidation

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## INTRODUCTION

The pelagic deposition of a sediment column is considered to have two stages which occur simultaneously. At the uppermost part the material is in a state of dispersion, which is governed by Kynch's theory of hindered settling. At the bottom of the column, soil is formed. This soil accretes because of the accumulation of new sediment and simultaneously consolidates under its self-weight.

Studies by Michaels & Bolger (1962) and by Been & Sills (1981) have shown the existence of a transition zone between dispersion and soil, which is characterized by large concentration gradients with depth. Been (1980) has demonstrated that consolidation and hindered settling derive from the same basic principles, and that hindered settling can be deduced from consolidation theory by setting the effective stress to be zero.

In this Note we derive an extended equation which governs the sedimentation and simultaneous consolidation of pelagic sediments. An example of the implementation of this theory is also given.

## HINDERED SETTLING

Kynch (1952) has derived a theory of sedimentation based on the fundamental assumption that the speed of fall of particles in a dispersion is determined only by the local particle density. The equation governing hindered settling is

$$V(c)\frac{\partial c}{\partial \xi}+\frac{\partial c}{\partial t}=0 \qquad (1)$$

where $c$ is the solids concentration, $\xi$ is the Eulerian co-ordinate and

$$V(c)=v_s+c\frac{dv_s}{dc} \qquad (2)$$

In equation (2) $v_s$ is the velocity of settling, which is a function of the local concentration:

$$v_s=v_s(c) \qquad (3)$$

Kynch's original analysis shows that several different modes of settling may occur, depending on the relationship between $v_s$ and $c$ and on the initial condition of the process. In particular, the theory shows that layers may exist in the dispersion where the value of the concentration changes abruptly, and that these discontinuities are responsible for linear and non-linear settling modes.

## CONSOLIDATION

Gibson, England & Hussey (1967) have derived the equation governing non-linear, finite strain consolidation of saturated clays. The theory is based on the assumption that the effective stress principle fully applies, i.e.

$$\sigma=\sigma'+u_w \qquad (4)$$

where $\sigma$ is the total stress, $\sigma'$ is the effective stress and $u_w$ is the total porewater pressure. The governing equation for monotonic, primary consolidation is

$$\left(\frac{\gamma_s}{\gamma_w}-1\right)\frac{d}{de}\left(\frac{k}{1+e}\right)\frac{\partial e}{\partial z}+\frac{\partial}{\partial z}\left[\frac{k}{\gamma_w(1+e)}\frac{d\sigma'}{de}\frac{\partial e}{\partial z}\right]+\frac{\partial e}{\partial t}=0 \qquad (5)$$

where $e$ is the void ratio, $\gamma_s$ and $\gamma_w$ are the unit weights of the solids and pore fluid respectively, $k$ is the coefficient of permeability, $t$ is the time and $z$ is a reduced co-ordinate measured against gravity. This co-ordinate is defined as the volume of solids (per unit area) lying between the datum plane and the Lagrangian (initial) co-ordinate point (Terzaghi, 1927; McNabb, 1960). The relationship between reduced, Lagrangian $a$ and Eulerian $\xi$ co-ordinates is given by

$$z=\int_0^a \frac{da'}{1+e(a',0)}=\int_0^\xi \frac{d\xi'}{1+e(\xi',t)} \qquad (6)$$

In particular, $h_z$ designates the total height of solids of the consolidating medium.

## LINKED THEORY

A more general form of the effective stress principle can be stated as

$$\sigma=\beta(e)\sigma'+u_w \qquad (7)$$

Discussion on this Technical Note closes on 1 July 1985. For further details see inside back cover.

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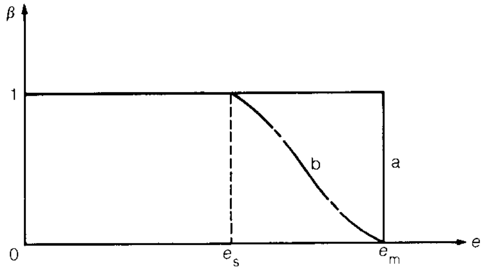


**Fig. 1. Forms of the β constitutive relationship**

where the interaction coefficient $\beta$ is a monotonic function of the void ratio. A reasonable form of the $(\beta, e)$ constitutive relationship of a soil–water mixture is shown in Fig. 1.

The interaction coefficient is equal to zero for values of the void ratio greater than $e_m$. In this case the particles (or aggregates of particles) are so distant that their interaction or contact is negligible and the mixture behaves as a dispersion. For values of the void ratio less than $e_s$ there is full particle-to-particle contact and the interaction coefficient is equal to unity, i.e. the conventional effective stress principle fully applies and the mixture behaves as a soil. Two kinds of $(\beta, e)$ constitutive relationships are shown in Fig. 1. In the general case b the interaction coefficient is a continuous function of the void ratio. In case a an abrupt change in $\beta$ occurs at $e_m$.

Assuming the validity of equation (7), the finite strain-governing equation expands to

$$\left(\frac{\gamma_s}{\gamma_w}-1\right)\frac{d}{de}\left(\frac{k}{1+e}\right)\frac{\partial e}{\partial z}+\frac{\partial}{\partial z}\left[\frac{k}{\gamma_w(1+e)}\beta\frac{d\sigma'}{de}\frac{\partial e}{\partial z}\right]$$
$$+\frac{\partial}{\partial z}\left[\frac{k}{\gamma_w(1+e)}\frac{d\beta}{de}\sigma'\frac{de}{dz}\right]+\frac{\partial e}{\partial t}=0 \quad (8)$$

In the following it is shown that this simple concept can explain, from a mechanical point of view, the behavior of soil–water mixtures in a wide range of solids concentration and that the governing equation (8) can model simultaneous sedimentation and consolidation of saturated soils.

For values of the void ratio greater than $e_m$ the soil–water mixture is truly a suspension. Effective stresses are absent, and the interaction coefficient, as well as its derivative $d\beta/de$, is equal to zero. In this case the governing equation (8) reduces to

$$\left(\frac{\gamma_s}{\gamma_w}-1\right)\frac{d}{de}\left(\frac{k}{1+e}\right)\frac{\partial e}{\partial z}+\frac{\partial e}{\partial t}=0 \quad e>e_m \quad (9)$$

or

$$V_z(e)\frac{\partial e}{\partial z}+\frac{\partial e}{\partial t}=0 \quad e>e_m \quad (10)$$

where

$$V_z(e)=\left(\frac{\gamma_s}{\gamma_w}-1\right)\frac{d}{de}\left(\frac{k}{1+e}\right) \quad (11)$$

It is noted that equation (10) has the same form as Kynch's equation (1), which governs hindered settling processes. From Darcy's law and the continuity equation it has been shown elsewhere (Been, 1980) that the relationship between $v_s$ in Kynch's theory and the coefficient of permeability is of the form

$$v_s=-\left(\frac{\gamma_s}{\gamma_w}-1\right)\frac{k}{1+e} \quad (12)$$

Thus for values of the void ratio greater than $e_m$ linear and/or non-linear settling modes occur and the behavior of the suspension is uniquely determined by the functional $\{k(e)\}$.

For values of the void ratio less than $e_s$ the soil–water mixture is truly a soil. Effective stresses are fully active and the interaction coefficient is equal to unity. In this case the extended governing equation (8) reduces to the Gibson–England–Hussey equation (5). The relationship between velocity of settling and coefficient of permeability expressed by equation (12) no longer applies and the behavior of the soil (i.e. the progress of consolidation) depends on the two functionals $\{k(e)\}$ and $\{\sigma'(e)\}$.

At intermediate concentrations, i.e. for values of the void ratio between $e_s$ and $e_m$, the soil–water mixture is no longer a suspension but is also not a soil. Pore fluid velocities are still high, and thus drag forces are comparable with effective stresses which start to become active. Floc orientation is very unstable and fluid paths change rapidly with time. Channeling is also likely to occur. However, in the macro sense, the extended governing equation (8) is still valid in this intermediate range.

## IMPLEMENTATION/APPLICATION

In the following we show the implementation of the proposed model in a practical application, i.e. the sedimentation and simultaneous consolidation of a soil–water mixture at uniform initial concentration.

The method of characteristics, used in conjunction with the reduced co-ordinates z, provides a simple tool toward the solution of the process. In fact this method furnishes the solution of the sedimentation mode and the domain

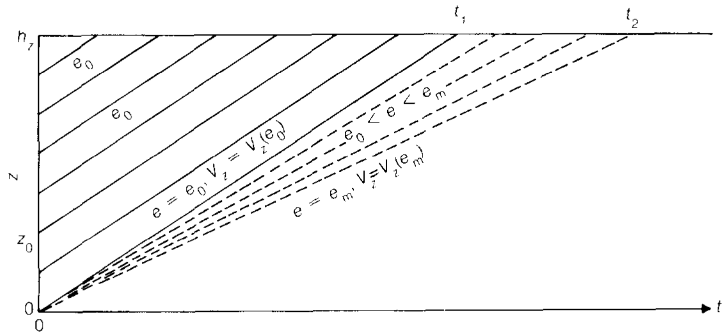


**Fig. 2. Relationship between height of solids and time**

of the consolidation mode in which equation (5) and/or equation (8) can be integrated by means of a numerical method. We start from equation (10) and seek the characteristics of this equation. By definition, on the characteristics

$$e(z+\mathrm{d}z, t+\mathrm{d}t) = e(z, t) \tag{13}$$

or

$$\frac{\partial e}{\partial z} = -\frac{\partial e}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}z} \tag{14}$$

From the system of equations (10) and (14) we obtain

$$\frac{\mathrm{d}z}{\mathrm{d}t} = V_z(e) \tag{15}$$

This equation (15) is the differential equation of the characteristic. Since along a characteristic $e$ (and thus $V_z(e)$) is constant, the characteristics of the governing equation (10) are straight lines of slope $V_z(e)$. The solution in the fixed $z$ domain is then uniquely determined as shown in Fig. 2. For simplicity of exposition it is assumed that the initial void ratio $e_0$ is constant with depth and that the $(\beta, e)$ constitutive relationship has the form shown in Fig. 1, case a.

If, in addition, the functional $\{V_z(e)\}$ given by equation (11) is a monotonically increasing function of the void ratio then the solution has no discontinuities.* The characteristics have continuously decreasing slopes as shown in Fig. 2, and non-linear hindered settling will occur.

Since the initial distribution is uniform ($e(z, 0) = e_0 = \text{constant}$), the characteristics in the triangle $0h_zt_1$ are parallel lines whose equations are given by

$$z = z_0 + V_z(e_0)t \tag{16}$$

where

$$z_0 = \int_0^a \frac{\mathrm{d}a'}{1+e(a', 0)} = \int_0^a \frac{\mathrm{d}a'}{1+e_0} \tag{17}$$

* More exactly, the solution has discontinuities of the first order, as defined by Kynch (1952) in his original paper.

At the datum plane ($z = 0$), i.e. at the bottom of the suspension, a characteristic of concentration $e_m$ is immediately propagated upward at the velocity $V_z(e_m)$. This soil formation line is represented by the line $0t_2$ in Fig. 2 and separates the soil–water mixture in two regions; in the triangle $0h_zt_2$ the mixture is a suspension. Equation (10) then applies and hindered settling occurs. Below the line $0t_2$, effective stresses are fully active and the consolidation equations (5) and/or (8) are valid. However, this does not result in a discontinuity of the dependent variable $e$. In this particular case, in the triangle $0t_1t_2$ the void ratio changes continuously from the initial value $e_0$ to the soil formation value $e_m$. In this region the solution depends on both the initial condition and the boundary condition of the convection equation (10) and the characteristics have equations given by

$$z = V_z(e)t \qquad e_0 < e < e_m \tag{18}$$

Moreover, the solution in the triangle $0h_zt_1$ depends only on the initial condition of equation (10) and the solution below the line $0t_2$ depends on the two boundary conditions of equation (5).

These considerations provide an insight to some general features of the process. In Fig. 2, the horizontal line $z = h_z$ represents the water–slurry interface of the suspension moving downward at a velocity which is a function of time. By using reduced $z$ co-ordinates this boundary does not change with time. The value of the void ratio at this boundary is equal to the initial value $e_0$, until time $t_1$ which is given by

$$t_1 = h_z/V_z(e_0) \tag{19}$$

Thus until $t_1$ the settling mode is linear.* After

* This does not mean that the velocity of fall of the interface is constant. Consolidation occurring below the line $0t_2$ will result in a non-linear curve of fall, which is ignored in Kynch's theory.

$t_1$, the soil–water interface can no longer maintain its concentration at the initial value. The value of the void ratio at this point $e_{top}$ will vary continuously between times $t_1$ and $t_2$ according to the equation

$$t = h_z/V_z(e_{top}) \quad t_1 < t < t_2 \quad e_0 < e_{top} < e_m \tag{20}$$

The time $t_2$ is given by

$$t_2 = h_z/V_z(e_m) \tag{21}$$

and until this time the settling mode will be non-linear. After time $t_2$ the characteristic $0t_2$ having a concentration equal to that of soil formation has reached the water–slurry interface and hindered settling is no longer possible. Only consolidation occurs at times greater than $t_2$.

The solution in the consolidation domain below the soil formation line $0t_2$ can be found by numerical integration of equations (5) and/or (8). A variety of numerical techniques (e.g. finite difference techniques, method of lines, space–time finite element techniques) are available and have been successfully employed.

From a physical point of view, it is more convenient to express the dependent variables in terms of Eulerian co-ordinates $\xi$ and time $t$. Therefore, once the solution has been obtained in the reduced $z$ co-ordinates system, it is converted to the Eulerian system by means of the co-ordinate transformation (6) which is rewritten as

$$\xi(t) = \int_0^z [1 + e(z', t)]\, dz' \tag{22}$$

With the limits of integration set at the datum plane ($z = 0$) and at the total height of solids $h_z$, equation (22) will naturally lead to the curve of the fall of the water-slurry interface $h(t)$ according to

$$h(t) = \int_0^{h_z} [1 + e(z', t)]\, dz' \tag{23}$$

It is noted that, if the initial void ratio distribution is not uniform or if the functional $\{V_z(e)\}$ is not a monotonically increasing function of the void ratio, discontinuities of the first order are likely to occur in the solution; they can still be handled by the method of characteristics. Furthermore, it is recalled that an 'intermediate region' of soil densities has been experimentally observed in laboratory sedimentation tests with remote-sensed density measurements (Been 1980; Been & Sills, 1981). This intermediate region, which is characterized by large changes in concentration with depth, has been associated with effective stresses which are not fully active (i.e. $0 < \beta < 1$).

This analysis shows that such a region can also exist at values of the void ratio at which effective stresses are not present at all, i.e. $\beta = 0$, and is responsible for non-linear settling modes. Thus a $(\beta, e)$ relationship of the kind shown in Fig. 1, case a, might be accepted without loss of generality in the mechanical phenomenon to be modeled. Finally, it is worth noting the advantage of using reduced co-ordinates. The moving boundary problem is reduced to a problem with fixed geometry thus facilitating both the analysis and the physical interpretation of the process with respect to Kynch's original work.

## CONCLUSIONS

This analysis sets a general theory which links pelagic sedimentation and consolidation as a single process governed by a modified effective stress equation. We have shown how this process can be analyzed by the method of characteristics. We leave the detailed numerical solutions and the laboratory verification of this new theory to a later paper.

## ACKNOWLEDGEMENTS

We are pleased to acknowledge the financial support provided by the US National Science Foundation. We are further pleased to acknowledge the stimulating and helpful comments by R. E. Gibson, H. Y. Ko, H. W. Olsen and V. Sunara.

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